

Some simplicial complexes of universal Osborn loops ^{*†}

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Abstract

A loop is shown to be a universal Osborn loop if and only if it has a particular simplicial complex. A loop is shown to be a universal Osborn loop and obeys two new identities if and only if it has another particular simplicial complex. A universal Osborn loop and four of its isotopes are shown to form a rectangular pyramid in a 3-dimensional space.

1 Introduction and Preliminaries

A loop is called an Osborn loop if it obeys any of the two identities below.

$$\text{OS}_3 : (x \cdot yz)x = xy \cdot [(x^\lambda \cdot xz) \cdot x] \quad (1)$$

$$\text{OS}_5 : (x \cdot yz)x = xy \cdot [(x \cdot x^\rho z) \cdot x] \quad (2)$$

For a comprehensive introduction to Osborn loops and its universality, and a detailed literature review on it, readers should check Jaiyéolá , Adéníran and Sòlárìn [3] and Jaiyéolá [4]. In this present paper, we shall follow the style and notations used in Jaiyéolá , Adéníran and Sòlárìn [3] and Jaiyéolá [4]. The only concepts and notions which will be introduced here are those that were not defined in Jaiyéolá , Adéníran and Sòlárìn [3] and Jaiyéolá [4].

Definition 1.1 *Let (L, \cdot) be a loop and $U, V, W \in \text{SYM}(L, \cdot)$.*

1. *If $(U, V, W) \in \text{AUT}(L, \cdot)$ for some V, W , then U is called autotopic.*
2. *If $(U, V, W) \in \text{AUT}(L, \cdot)$ such that $W = U, V = I$, then U is called λ -regular.*
3. *If $(U, V, W) \in \text{AUT}(L, \cdot)$ such that $U = I, W = V$, then V is called ρ -regular.*

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Drisko [2] while considering the action of isotopisms and autotopisms of loops, found it convenient to think of a loop $\mathcal{Q} = (Q, \cdot, \backslash, /)$ in terms of the set $T_{\mathcal{Q}}$ of all ordered triples (x, y, z) of elements of Q such that $x \cdot y = z$. An isotopism (α, β, γ) from G to H takes $(x, y, z) \in T_G$ to $(x\alpha, y\beta, z\gamma) \in T_H$. We shall adopt his conventions at some points in time. We shall denote by $[\alpha, \beta]$, the commutator of any $\alpha, \beta \in \text{SYM}(G, \cdot)$.

Let $(Q, \cdot, \backslash, /)$ be a loop, then we shall be making use of the following notations for principal isotopes of (Q, \cdot) .

- $(Q, *_0)$ represents $Q_{x,v}$;
- (Q, \circ_0) represents $Q_{u, \phi_0(x, u, v)}$, $\phi_0(x, u, v) = (u \backslash ((uv)/(u \backslash (xv)))v)$;
- (Q, \circ_1) represents $Q_{u, [u \backslash (xv)]}$;
- $(Q, *_1)$ represents $Q_{\phi_1(x, u, v), v}$, $\phi_1(x, u, v) = (u \backslash ((uv)/(u \backslash (xv)))v)$ for all $x, u, v \in Q$;
- (Q, \circ_2) represents $Q_{x, \phi_2(x, u, v)}$, $\phi_2(x, u, v) = (u \backslash [(u/v)(u \backslash (xv))])$;
- (Q, \circ_3) represents $Q_{[x \cdot u \backslash v]/v, [u \backslash (xv)]}$;
- $(Q, *_2)$ represents $Q_{u, e}$;
- $(Q, *_3)$ represents $Q_{e, v}$.

Let (G, \cdot) be a loop and let

$$BS_2(G, \cdot) = \{\theta \in \text{SYM}(G) : G(a, b) \stackrel{\theta}{\cong} G(c, d) \text{ for some } a, b, c, d \in G\}.$$

As shown in Bryant and Schneider [1], $BS_2(G, \cdot)$ forms a group for a loop (G, \cdot) and it shall be called the second Bryant-Schneider group (2^{nd} BSG) of the loop.

Consider the following two notions in algebraic topology.

Definition 1.2 Let V_Q be a set of isotopes of a loop (Q, \cdot) and let $S_Q \subseteq \mathcal{Q}^{V_Q}$ such that $\phi \in S_Q$. If S_Q is a topology on V_Q , then it is called the topology of isotopes of the loop Q and the pair (V_Q, S_Q) is called a topological space of isotopes of Q if (V_Q, S_Q) is a topological space.

Based on the above notion of topological space of isotopes of a loop, the following facts are direct consequences.

Lemma 1.1 Let (Q, \cdot) be a loop and let V_Q be the set of isotopes of Q . Then, $\left(V_Q, \mathcal{Q}^{V_Q}\right)$ is a topological space of isotopes of Q .

Lemma 1.2 Let (Q, \cdot) be a G -loop and let V_Q be the set of isotopes of Q . Let $S_Q = \{X_i\}_{i \in \Omega} \subseteq \mathcal{Q}^{V_Q}$ such that $\phi \in S_Q$ and $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. Then, (V_Q, S_Q) is a topological space of isotopes of Q .

Corollary 1.1 *Let (Q, \cdot) be a CC-loop or VD-loop or K-loop or Buchsteiner loop or extra loop or group. Let $S_Q = \{X_i\}_{i \in \Omega} \subseteq \mathcal{Z}^{V_Q}$ such that $\phi \in S_Q$ and $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. Then, (V_Q, S_Q) is a topological space of isotopes of Q .*

Definition 1.3 *A simplicial complex is a pair (V, S) where V is a set of points called vertices and S is a given family of finite subsets, called simplexes, so that the following conditions are satisfied:*

1. *all points of V are simplexes;*
2. *any non-empty subset of a simplex is a simplex.*

A simplex consisting of $(n + 1)$ points is called n -dimensional simplex.

Definition 1.4 *Let V_Q be a set of isotopes of a loop (Q, \cdot) and let $S_Q \subseteq \mathcal{Z}^{V_Q}$. If $K_Q = (V_Q, S_Q)$ is a simplicial complex, then K_Q is called a trivial simplicial complex of isotopes of the loop Q .*

Definition 1.5 *Let V_Q be a set of isotopes of a loop (Q, \cdot) and let $S_Q = \{X_i\}_{i \in \Omega} \subseteq \mathcal{Z}^{V_Q}$ such that $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. If $K_Q = (V_Q, S_Q)$ is a simplicial complex, then K_Q is called a non-trivial simplicial complex of isotopes or simplicial complex of isotopes of the loop Q .*

The facts below follow suite.

Lemma 1.3 *Let (Q, \cdot) be a loop and let V_Q be the set of isotopes of Q . Then, $\left(V_Q, \mathcal{Z}^{V_Q}\right)$ is a trivial simplicial complex of isotopes of Q .*

Lemma 1.4 *Let (Q, \cdot) be a G -loop and let V_Q be the set of isotopes of Q . Let $S_Q = \{X_i\}_{i \in \Omega} \subseteq \mathcal{Z}^{V_Q}$ such that $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. Then, (V_Q, S_Q) is a simplicial complex of isotopes of Q .*

Corollary 1.2 *Let (Q, \cdot) be a CC-loop or VD-loop or K-loop or Buchsteiner loop or extra loop or group. Let $S_Q = \{X_i\}_{i \in \Omega} \subseteq \mathcal{Z}^{V_Q}$ such that $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. Then, (V_Q, S_Q) is a simplicial complex of isotopes of Q .*

Definition 1.6 *Let $K = (V, S)$ and $K' = (V', S')$ be two simplicial complexes. A simplicial map $f : K \rightarrow K'$ is a set map $f : V \rightarrow V'$ satisfying the property: for every simplex $x \in S$, the image $f(x) \in S'$.*

In this work, the notion of simplicial complex is used to characterize universal Osborn loops. The following results are important for the set objective.

Theorem 1.1 (Jaiyéplá , Adéníran and Sòlárìn [3])

Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop and $\gamma_0(x, u, v) = \mathbb{R}_v R_{[u \setminus (xv)]} \mathbb{L}_u L_x$ for all $x, u, v \in Q$, then \mathcal{Q} is a universal Osborn loop if and only if the commutative diagram

$$\begin{array}{ccc}
 & & (Q, \circ_0) \\
 & \nearrow^{(R_{\phi_0(x, u, v)}, L_u, I)} & \downarrow (\gamma_0, \gamma_0, \gamma_0) \text{ isomorphism} \\
 (Q, \cdot) & \xrightarrow[\text{principal isotopism}]{(R_v, L_x, I)} & (Q, *_0)
 \end{array} \tag{3}$$

holds.

Theorem 1.2 (Jaiyéplá [4])

Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop and $\gamma_1(x, u, v) = \mathbb{R}_v R_{[u \setminus (xv)]} \mathbb{L}_u L_x$ for all $x, u, v \in Q$, then \mathcal{Q} is a universal Osborn loop if and only if the commutative diagram

$$\begin{array}{ccc}
 & & (Q, *_1) \\
 & \nearrow^{(R_v, L_{\phi_1(x, u, v)}, I)} & \downarrow (\gamma_1, \gamma_1, \gamma_1) \text{ isomorphism} \\
 (Q, \cdot) & \xrightarrow[\text{principal isotopism}]{(R_{[u \setminus (xv)]}, L_u, I)} & (Q, \circ_1)
 \end{array} \tag{4}$$

holds.

Theorem 1.3 (Jaiyéplá , Adéníran and Sòlárìn [3])

Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop and $\gamma_0(x, u, v) = \mathbb{R}_v R_{[u \setminus (xv)]} \mathbb{L}_u L_x$ for all $x, u, v \in Q$, then \mathcal{Q} is a universal Osborn loop implies the commutative diagram

$$\begin{array}{ccc}
 & & (Q, \circ_2) \\
 & \nearrow^{(R_{\phi_2(x, u, v)}, L_x, I)} & \uparrow (\gamma_0, \gamma_0, \gamma_0) \text{ isomorphism} \\
 (Q, \cdot) & \xrightarrow[\text{principal isotopism}]{(I, L_u, I)} & (Q, *_2)
 \end{array} \tag{5}$$

holds.

Theorem 1.4 (Jaiyéplá [4])

Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop and $\gamma_1(x, u, v) = \mathbb{R}_v R_{[u \setminus (xv)]} \mathbb{L}_u L_x$ for all $x, u, v \in Q$, then \mathcal{Q} is a universal Osborn loop implies the commutative diagram

$$\begin{array}{ccc}
 & & (Q, \circ_3) \\
 & \nearrow^{(R_{[u \setminus (xv)]}, L_{[x \cdot u \setminus v]/v}, I)} & \uparrow (\gamma_1, \gamma_1, \gamma_1) \text{ isomorphism} \\
 (Q, \cdot) & \xrightarrow[\text{principal isotopism}]{(R_v, I, I)} & (Q, *_3)
 \end{array} \tag{6}$$

holds.

Lemma 1.5 (*Drisko [2]*)

Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop. Then $Q_{f,g} \cong Q_{c,d}$ if and only if there exists $(\alpha, \beta, \gamma) \in \text{AUT}(\mathcal{Q})$ such that $(f, g, fg)(\alpha, \beta, \gamma) = (c, d, cd)$.

Theorem 1.5 (*Bryant and Schneider [1]*)

Let $(Q, \cdot, \backslash, /)$ be a quasigroup. If $Q_{a,b} \stackrel{I}{\cong} Q_{c,d}$ if and only if $c \cdot b, a \cdot d \in N_\mu(Q_{a,b})$ and $a \cdot b = c \cdot d$.

2 Main Results

Theorem 2.1 Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then, the following are necessary and sufficient for each other.

1. $(Q, \circ_0) \stackrel{I}{\cong} (Q, \circ_1)$.
2. $(Q, *_0) \stackrel{I}{\cong} (Q, *_1)$.
3. \mathcal{Q} is a boolean group.

Proof

By combining the commutative diagrams in Equation 3 and Equation 4, we have the commutative diagram below.

$$\begin{array}{ccccc}
 & & (Q, \circ_1) & & \\
 & & \uparrow & & \\
 & & (Q, \circ_0) & & \\
 & \nearrow^{(R_{\phi_0}, L_u, I)} & \uparrow^{\gamma_{01}^\circ} & \searrow_{\gamma_0} & \\
 (Q, \cdot) & \xrightarrow{(R_v, L_x, I)} & (Q, *_0) & & \\
 \downarrow_{(R_v, L_{\phi_1}, I)} & & \searrow_{\gamma_{01}^*} & & \\
 (Q, *_1) & & & & (Q, *_1)
 \end{array}
 \tag{7}$$

Let

$$(Q, \circ_0) \xrightarrow[\text{isotopism}]{(\delta_{01}^\circ, \varepsilon_{01}^\circ, \pi_{01}^\circ)} (Q, \circ_1).$$

So, from Equation 7,

$$\begin{aligned}
(R_{\phi_0(x,u,v)}, L_u, I)(\delta_{01}^\circ, \varepsilon_{01}^\circ, \pi_{01}^\circ) &= (R_{[u \setminus (xv)]}, L_u, I) \Rightarrow \\
(R_{\phi_0(x,u,v)}\delta_{01}^\circ, L_u\varepsilon_{01}^\circ, \pi_{01}^\circ) &= (R_{[u \setminus (xv)]}, L_u, I) \Leftrightarrow \\
R_{\phi_0(x,u,v)}\delta_{01}^\circ &= R_{[u \setminus (xv)]}, L_u\varepsilon_{01}^\circ = L_u \text{ and } \pi_{01}^\circ = I \Leftrightarrow \\
\delta_{01}^\circ &= R_{\phi_0(x,u,v)}^{-1}R_{[u \setminus (xv)]}, \varepsilon_{01}^\circ = L_u^{-1}L_u = I \text{ and } \pi_{01}^\circ = I.
\end{aligned}$$

Thus, $(Q, \circ_0) \cong (Q, \circ_1)$ iff $\delta_{01}^\circ = \varepsilon_{01}^\circ = I$ iff

$$\begin{aligned}
R_{\phi_0(x,u,v)}^{-1}R_{[u \setminus (xv)]} &= I \Leftrightarrow \phi_0(x, u, v) = [u \setminus (xv)] \\
(u \setminus [(uv)/(u \setminus (xv))]v) &= [u \setminus (xv)] \Leftrightarrow x \setminus (uv) = u \setminus (xv).
\end{aligned}$$

Similarly, by using the procedure above, it can be shown that $(Q, *_0) \cong (Q, *_1)$ iff $x \setminus (uv) = u \setminus (xv)$.

Keeping in mind that every Osborn loop of exponent 2 is an abelian group, hence, a Boolean group. This completes the proof.

Remark 2.1 *It can be observed that in a universal Osborn loop $\mathcal{Q} = (Q, \cdot, \setminus, /)$ and for $\gamma_0(x, u, v)$ and $\gamma_1(x, u, v)$ of Theorem 1.1 and Theorem 1.2, $\gamma_0(x, u, v) = \gamma_1(x, u, v)$ if and only if $[\mathbb{L}_u L_x, \mathbb{R}_v R_{[u \setminus (xv)]}] = I$ for all $x, u, v \in Q$.*

The proof of Theorem 2.1 can also be achieved by making use of Theorem 1.5. Take $a = u$, $b = \phi_0(x, u, v)$, $c = u$ and $d = u \setminus (xv)$. Then, $(Q, \circ_0) \stackrel{I}{\cong} (Q, \circ_1)$ iff

(i) $u\phi_0(x, u, v) \in N_\mu((Q, \circ_0))$, (ii) $u[u \setminus (xv)] \in N_\mu((Q, \circ_0))$, (iii) $u\phi_0(x, u, v) = u[u \setminus (xv)] \Leftrightarrow \mathcal{Q}$ is a Boolean group.

Theorem 2.2 *Let $\mathcal{Q} = (Q, \cdot, \setminus, /)$ be a universal Osborn loop. Then $(Q, \circ_0) \cong (Q, \circ_1)$ if and only if there exists $(I, \beta, \gamma) \in \text{AUT}(\mathcal{Q})$ such that*

$$uv = xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \cdot xR_v\mathbb{L}_u = xR_v\gamma^{-1}\mathbb{R}_v \cdot xR_v\mathbb{L}_u \quad (8)$$

for all $x, u, v \in Q$.

Proof

Following Lemma 1.5, $(Q, \circ_0) \cong (Q, \circ_1)$ if and only if there exists $(\alpha, \beta, \gamma) \in \text{AUT}(\mathcal{Q})$ such that

$$\begin{aligned}
(u, \phi_0(x, u, v), u\phi_0(x, u, v))(\alpha, \beta, \gamma) &= (u, [u \setminus (xv)], xv) \Leftrightarrow \\
(u\alpha, \phi_0(x, u, v)\beta, (u\phi_0(x, u, v))\gamma) &= (u, [u \setminus (xv)], xv) \Leftrightarrow \\
u\alpha = u, \phi_0(x, u, v)\beta &= [u \setminus (xv)] \text{ and } (u\phi_0(x, u, v))\gamma = xv \Leftrightarrow \\
\alpha = I, \{u \setminus [(uv)/(u \setminus (xv))]v\}\beta &= u \setminus (xv) \text{ and } \{[(uv)/(u \setminus (xv))]v\}\gamma = xv \Leftrightarrow \\
\alpha = I, [(uv)/(u \setminus (xv))]R_v\mathbb{L}_u\beta &= xR_v\mathbb{L}_u \text{ and } [(uv)/(u \setminus (xv))]R_v\gamma = xR_v \Leftrightarrow \\
\alpha = I, (uv)/(u \setminus (xv)) &= xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \text{ and } [(uv)/(u \setminus (xv))] = xR_v\gamma^{-1}\mathbb{R}_v \Leftrightarrow \\
\alpha = I, uv = xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \cdot xR_v\mathbb{L}_u &\text{ and } uv = xR_v\gamma^{-1}\mathbb{R}_v \cdot xR_v\mathbb{L}_u \Leftrightarrow
\end{aligned}$$

there exists $(I, \beta, \gamma) \in \text{AUT}(\mathcal{Q})$ such that

$$uv = xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \cdot xR_v\mathbb{L}_u = xR_v\gamma^{-1}\mathbb{R}_v \cdot xR_v\mathbb{L}_u.$$

Remark 2.2 *If the autotopism (α, β, γ) in Theorem 2.2 is the identity autotopism, then we shall have the equivalence of 1. and 3. of Theorem 2.1.*

Corollary 2.1 *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, \circ_0) \cong (Q, \circ_1)$ implies that there exists $(I, \beta, \gamma) \in \text{AUT}(\mathcal{Q})$ such that $\gamma = \mathbb{L}_u\beta L_u$ for all $u \in Q$. Hence,*

1. $\gamma = \beta$ iff $[\beta, L_u] = I$ or $[\gamma, L_u] = I$. Thence, β is a ρ -regular permutation.
2. $\gamma = L_u$ iff $\beta = L_u$. Thence, \mathcal{Q} is an abelian group.

Proof

The proof of these follows from the fact in Theorem 2.2 that

$$xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \cdot xR_v\mathbb{L}_u = xR_v\gamma^{-1}\mathbb{R}_v \cdot xR_v\mathbb{L}_u \Rightarrow$$

$\mathbb{L}_u\beta L_u = \gamma$ for all $u \in Q$.

Theorem 2.3 *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, *_0) \cong (Q, *_1)$ if and only if there exists $(\delta, I, \pi) \in \text{AUT}(\mathcal{Q})$ such that*

$$uv = x \cdot x\delta R_v\mathbb{L}_u = x \cdot xR_v\pi\mathbb{L}_u \tag{9}$$

for all $x, u, v \in Q$.

Proof

Following Lemma 1.5, $(Q, *_0) \cong (Q, *_1)$ if and only if there exists $(\delta, \varepsilon, \pi) \in \text{AUT}(\mathcal{Q})$ such that $(x, v, xv)(\delta, \varepsilon, \pi) = (\phi_1(x, u, v), v, \phi_1(x, u, v)v)$. The procedure of the proof of the remaining part is similar to that of Theorem 2.2.

Remark 2.3 *If the autotopism $(\delta, \varepsilon, \pi)$ in Theorem 2.3 is the identity autotopism, then we shall have the equivalence of 2. and 3. of Theorem 2.1.*

Corollary 2.2 *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, *_0) \cong (Q, *_1)$ implies that there exists $(\delta, I, \pi) \in \text{AUT}(\mathcal{Q})$ such that $\pi = \mathbb{R}_v\delta R_v$ for all $v \in Q$. Hence,*

1. $\pi = \delta$ iff $[\delta, R_v] = I$ or $[\pi, R_v] = I$. Thence, δ is a λ -regular permutation.
2. $\delta = R_v$ iff $\pi = R_v$. Thence, \mathcal{Q} is an abelian group.

Proof

The proof of these follows from the fact in Theorem 2.3 that

$$x \cdot x\delta R_v\mathbb{L}_u = x \cdot xR_v\pi\mathbb{L}_u \Rightarrow$$

$\pi = \mathbb{R}_v\delta R_v$ for all $v \in Q$.

Theorem 2.4 Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, \circ_0) \cong (Q, \circ_1)$ and $(Q, *_0) \cong (Q, *_1)$ if and only if there exists $(I, \beta, \gamma), (\delta, I, \pi) \in \text{AUT}(\mathcal{Q})$ such that

$$uv = xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \cdot xR_v\mathbb{L}_u = xR_v\gamma^{-1}\mathbb{R}_v \cdot xR_v\mathbb{L}_u = x \cdot x\delta R_v\mathbb{L}_u = x \cdot xR_v\pi\mathbb{L}_u \quad (10)$$

for all $x, u, v \in Q$

Proof

This is achieved by simply combining Theorem 2.2 and Theorem 2.3.

Theorem 2.5 Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. If $(Q, \circ_0) \stackrel{\gamma_{01}^\circ}{\cong} (Q, \circ_1)$ and $(Q, *_0) \stackrel{\gamma_{01}^*}{\cong} (Q, *_1)$, then $\gamma_0\gamma_{01}^*\gamma_1 = \gamma_{01}^\circ$.

Proof

The commutative diagram in Equation 7 proves this.

Corollary 2.3 Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. If $(Q, \circ_0) \cong (Q, \circ_1)$ and $(Q, *_0) \cong (Q, *_1)$, then the following are necessary and sufficient for each other.

1. $\beta = I$.
2. $\gamma = I$.
3. $\delta = I$.
4. $\pi = I$.
5. $(Q, \circ_0) \stackrel{I}{\cong} (Q, \circ_1)$.
6. $(Q, *_0) \stackrel{I}{\cong} (Q, *_1)$.
7. \mathcal{Q} is a boolean group.

Proof

To prove the equivalence of 1. to 4. and 7., use Equation 10 of Theorem 2.4. The proof of the equivalence of 5. to 7. follows from Theorem 2.1.

Remark 2.4 Corollary 2.3 is a very important result in this study. It gives us the main distinctions between Theorem 2.1 and Theorem 2.4. That is, the necessary and sufficient condition(s) under which the isomorphisms $(Q, \circ_0) \cong (Q, \circ_1)$ and $(Q, *_0) \cong (Q, *_1)$ will be trivial. And the condition(s) is when any of the autotopic permutations of β, γ, δ and π of Theorem 2.2 and Theorem 2.3 is equal to the identity mapping.

Next, it is important to deduce the actual definitions of the autotopic mappings $\beta, \gamma, \delta, \pi$ and the isomorphisms γ_{01}^* and γ_{01}° . Recall that by the necessary part of Lemma 1.5, if $\mathcal{Q} = (Q, \cdot, \backslash, /)$ is a loop and $Q_{f,g} \stackrel{\theta}{\cong} Q_{c,d}$, then there exists $(A, B, C) \in \text{AUT}(\mathcal{Q})$ such that $(f, g, fg)(A, B, C) = (c, d, cd)$. According to the proof of this,

$$(A, B, C) = (R_g\theta R_d^{-1}, L_f\theta L_c^{-1}, \theta) \Leftrightarrow A = R_g\theta R_d^{-1}, B = L_f\theta L_c^{-1} \text{ and } C = \theta. \quad (11)$$

Thus,

$$I = \alpha = R_{\phi_o(x,u,v)}\gamma_{01}^\circ R_{[u \setminus (xv)]}^{-1}, \beta = L_u\gamma_{01}^\circ L_u^{-1} \text{ and } \gamma = \gamma_{01}^\circ$$

$$\gamma_{01}^\circ = \mathbb{R}_{\phi_o(x,u,v)}R_{[u \setminus (xv)]}, \beta = L_u\mathbb{R}_{\phi_o(x,u,v)}R_{[u \setminus (xv)]}\mathbb{L}_u^{-1} \text{ and } \gamma = \mathbb{R}_{\phi_o(x,u,v)}R_{[u \setminus (xv)]}$$

and

$$\begin{aligned}\delta &= R_v \gamma_{01}^* R_v^{-1}, \quad I = \varepsilon = L_x \gamma_{01}^* L_{\phi_1(x,u,v)}^{-1} \text{ and } \pi = \gamma_{01}^* \\ \delta &= R_v \gamma_{01}^* \mathbb{R}_v^{-1}, \quad \gamma_{01}^* = \mathbb{L}_x L_{\phi_1(x,u,v)} \text{ and } \pi = \gamma_{01}^* \\ \delta &= R_v \mathbb{L}_x L_{\phi_1(x,u,v)} \mathbb{R}_v^{-1}, \quad \gamma_{01}^* = \mathbb{L}_x L_{\phi_1(x,u,v)} \text{ and } \pi = \mathbb{L}_x L_{\phi_1(x,u,v)}.\end{aligned}$$

Therefore, Theorem 2.2 and Theorem 2.3 can now be restated as follows.

Theorem 2.6 *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, \circ_0) \stackrel{\gamma_{01}^\circ}{\cong} (Q, \circ_1)$ if and only if*

$$y \cdot u \backslash [(uz)\psi_0] = (yz)\psi_0 \text{ and } uv = xR_v(R_v\psi_0)^{-1} \cdot xR_v\mathbb{L}_u \quad (12)$$

where $\psi_0 = \mathbb{R}_{\phi_0(x,u,v)} R_{[u \backslash (xv)]}$ for all $x, y, z, u, v \in Q$

Proof

Simply substitute

$$\beta = L_u \mathbb{R}_{\phi_0(x,u,v)} R_{[u \backslash (xv)]} \mathbb{L}_u^{-1} \text{ and } \gamma = \mathbb{R}_{\phi_0(x,u,v)} R_{[u \backslash (xv)]}$$

into Equation 8 of Theorem 2.2.

Theorem 2.7 *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, *_0) \stackrel{\gamma_{01}^*}{\cong} (Q, *_1)$ if and only if*

$$[(yv)\psi_1]/v \cdot z = (yz)\psi_1 \text{ and } uv = x \cdot u \backslash [(xv)\psi_1] \quad (13)$$

where $\psi_1 = \mathbb{L}_x L_{\phi_1(x,u,v)}$ for all $x, y, z, u, v \in Q$

Proof

Simply substitute

$$\delta = R_v \mathbb{L}_x L_{\phi_1(x,u,v)} \mathbb{R}_v^{-1} \text{ and } \pi = \mathbb{L}_x L_{\phi_1(x,u,v)}$$

into Equation 9 of Theorem 2.3.

Lemma 2.1 *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop.*

1. \mathcal{Q} is a universal Osborn loop and obeys Equation 12 if and only if $\gamma_0, \gamma_{01}^\circ \in BS_2(\mathcal{Q})$.
2. \mathcal{Q} is a universal Osborn loop and obeys Equation 13 if and only if $\gamma_1, \gamma_{01}^* \in BS_2(\mathcal{Q})$.

Proof

This follows by combining Theorem 1.1, Theorem 1.2, Theorem 2.2 and Theorem 2.3

Remark 2.5 *It is a self exercise to confirm if $(Q, \circ_0) \stackrel{\gamma_{01}^\circ}{\cong} (Q, \circ_1)$ and $(Q, *_0) \stackrel{\gamma_{01}^*}{\cong} (Q, *_1)$ in some universal Osborn loops like Moufang loops and extra loops by simply verifying Equation 12 and Equation 13. Furthermore, the relation $\gamma_0 \gamma_{01}^* \gamma_1 = \gamma_{01}^\circ$ of Theorem 2.5 is justifiable as well. It must be noted also, that in any universal Osborn loop \mathcal{Q} , Equation 12 and Equation 13 are necessary and sufficient conditions for $\gamma_{01}^*, \gamma_{01}^\circ \in BS_2(\mathcal{Q})$.*

By combining the commutative diagrams in Equation 5 and Equation 6, we have the commutative diagram below.

$$\begin{array}{ccccc}
 & & (Q, \circ_3) & & \\
 & & \uparrow & & \\
 & & \gamma_{23}^\circ & & \\
 & & (Q, \circ_2) & & \\
 & \nearrow (R_{\phi_2}, L_x, I) & \searrow \gamma_0 & & \\
 (Q, \cdot) & \xrightarrow{(I, L_u, I)} & (Q, *_2) & & \\
 \downarrow (R_v, I, I) & & \searrow \gamma_{23}^* & & \\
 & & (Q, *_3) & & \\
 & & \uparrow \gamma_1 & & \\
 & & (Q, \circ_3) & &
 \end{array} \tag{14}$$

Theorem 2.8 Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, \circ_2) \cong (Q, \circ_3)$ if and only if there exists $(\lambda, \mu, \nu) \in \text{AUT}(\mathcal{Q})$ such that

$$\lambda = R_{u \backslash v} \mathbb{R}_v, \quad \mu = L_u \mathbb{L}_{u \backslash v} \quad \text{and} \quad [x \cdot x R_v \mathbb{L}_u \mu^{-1}] \nu = x \lambda \cdot x R_v \mathbb{L}_u \tag{15}$$

for all $x, u, v \in Q$.

Proof

Following Lemma 1.5, $(Q, \circ_2) \cong (Q, \circ_3)$ if and only if there exists $(\lambda, \mu, \nu) \in \text{AUT}(\mathcal{Q})$ such that $(x, \phi_2(x, u, v), x \phi_2(x, u, v))(\lambda, \mu, \nu) = ([x \cdot u \backslash v]/v, [u \backslash (xv)], \{[x \cdot u \backslash v]/v\}[u \backslash (xv)])$. The procedure of the proof of the remaining part is similar to that of Theorem 2.2.

Lemma 2.2 Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, \circ_2) \overset{\gamma_{23}^\circ}{\cong} (Q, \circ_3)$ if and only if there exists $(\lambda, \mu, \gamma_{23}^\circ) \in \text{AUT}(\mathcal{Q})$ such that

$$\gamma_{23}^\circ = \mathbb{R}_{\phi_2(x, u, v)} R_{u \backslash v} \mathbb{R}_v R_{[u \backslash (xv)]} = \mathbb{L}_x L_u \mathbb{L}_{u \backslash v} L_{\{[x \cdot u \backslash v]/v\}} \quad \text{and} \quad [x \cdot x R_v \mathbb{L}_u \mu^{-1}] \gamma_{23}^\circ = x \lambda \cdot x R_v \mathbb{L}_u \tag{16}$$

for all $x, u, v \in Q$.

Proof

Considering the commutative diagram in Equation 14 and using Equation 11,

$$\lambda = R_{\phi_2(x, u, v)} \gamma_{23}^\circ R_{[u \backslash (xv)]}^{-1}, \quad \mu = L_x \gamma_{23}^\circ L_{\{[x \cdot u \backslash v]/v\}}^{-1} \quad \text{and} \quad \nu = \gamma_{23}^\circ.$$

The final conclusion follows from Theorem 2.8.

Corollary 2.4 *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. $\gamma_{23}^\circ \in BS_2(\mathcal{Q})$ if and only if there exists $(\lambda, \mu, \gamma_{23}^\circ) \in AUT(\mathcal{Q})$ such that*

$$\gamma_{23}^\circ = \mathbb{R}_{\phi_2(x,u,v)} R_{u \backslash v} \mathbb{R}_v R_{[u \backslash (xv)]} = \mathbb{L}_x L_u \mathbb{L}_{u \backslash v} L_{\{[x \cdot u \backslash v]/v\}} \text{ and } [x \cdot x R_v \mathbb{L}_u \mu^{-1}] \gamma_{23}^\circ = x \lambda \cdot x R_v \mathbb{L}_u \quad (17)$$

for all $x, u, v \in Q$.

Proof

This follows from Lemma 2.2.

Corollary 2.5 *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop. \mathcal{Q} is a universal Osborn loop and $\gamma_{23}^\circ \in BS_2(\mathcal{Q})$ implies $\gamma_0 \in BS_2(\mathcal{Q})$ and there exists $(\lambda, \mu, \gamma_{23}^\circ) \in AUT(\mathcal{Q})$ such that*

$$\gamma_{23}^\circ = \mathbb{R}_{\phi_2(x,u,v)} R_{u \backslash v} \mathbb{R}_v R_{[u \backslash (xv)]} = \mathbb{L}_x L_u \mathbb{L}_{u \backslash v} L_{\{[x \cdot u \backslash v]/v\}} \text{ and } [x \cdot x R_v \mathbb{L}_u \mu^{-1}] \gamma_{23}^\circ = x \lambda \cdot x R_v \mathbb{L}_u \quad (18)$$

for all $x, u, v \in Q$.

Proof

This follows from Theorem 1.3 and Lemma 2.2.

Simplicial Complex of Isotopes of a Universal Osborn Loop

Theorem 2.9 *Let (Q, \cdot) be a loop. Let $V_0(Q) = \{(Q, \cdot), (Q, \circ_0), (Q, *_0)\}$ and $S_0(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_0)\}, \{(Q, *_0)\}, \{(Q, \circ_0), (Q, *_0)\}\}$. Then, (Q, \cdot) is a universal Osborn loop if and only if $K_0(Q) = (V_0(Q), S_0(Q))$ is a simplicial complex of isotopes of (Q, \cdot) .*

Proof

This is proved with the help of Theorem 1.1.

Theorem 2.10 *Let (Q, \cdot) be a loop. Let $V_1(Q) = \{(Q, \cdot), (Q, \circ_1), (Q, *_1)\}$ and $S_1(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_1)\}, \{(Q, *_1)\}, \{(Q, \circ_1), (Q, *_1)\}\}$. Then, (Q, \cdot) is a universal Osborn loop if and only if $K_1(Q) = (V_1(Q), S_1(Q))$ is a simplicial complex of isotopes of (Q, \cdot) .*

Proof

This is proved with the help of Theorem 1.2.

Theorem 2.11 *Let (Q, \cdot) be a loop. Let $V_2(Q) = \{(Q, \cdot), (Q, \circ_2), (Q, *_2)\}$ and $S_2(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_2)\}, \{(Q, *_2)\}, \{(Q, \circ_2), (Q, *_2)\}\}$. If (Q, \cdot) is a universal Osborn loop, then $K_2(Q) = (V_2(Q), S_2(Q))$ is a simplicial complex of isotopes of (Q, \cdot) .*

Proof

This is proved with Theorem 1.3.

Theorem 2.12 *Let (Q, \cdot) be a loop. Let $V_3(Q) = \{(Q, \cdot), (Q, \circ_3), (Q, *_3)\}$ and $S_3(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_3)\}, \{(Q, *_3)\}, \{(Q, \circ_3), (Q, *_3)\}\}$. If (Q, \cdot) is a universal Osborn loop, then $K_3(Q) = (V_3(Q), S_3(Q))$ is a simplicial complex of isotopes of (Q, \cdot) .*

Proof

This is proved with the aid of Theorem 1.4.

Corollary 2.6 *Let (Q, \cdot) be a loop. Let $V_i(Q) = \{(Q, \cdot), (Q, \circ_i), (Q, *_i)\}$ and $S_i(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_i)\}, \{(Q, *_i)\}, \{(Q, \circ_i), (Q, *_i)\}\}$ for $i = 0, 1$. Then, (Q, \cdot) is a universal Osborn loop if and only if $K_{01}(Q) = K_0(Q) \cup K_1(Q) = (V_0(Q) \cup V_1(Q), S_0(Q) \cup S_1(Q))$ is a simplicial complex of isotopes of (Q, \cdot) .*

Proof

This follows from Theorem 2.9 and Theorem 2.10.

Corollary 2.7 *Let (Q, \cdot) be a loop. Let $V_i(Q) = \{(Q, \cdot), (Q, \circ_i), (Q, *_i)\}$ and $S_i(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_i)\}, \{(Q, *_i)\}, \{(Q, \circ_i), (Q, *_i)\}\}$ for $i = 2, 3$. If (Q, \cdot) is a universal Osborn loop, then $K_{23}(Q) = K_2(Q) \cup K_3(Q) = (V_2(Q) \cup V_3(Q), S_2(Q) \cup S_3(Q))$ is a simplicial complex of isotopes of (Q, \cdot) .*

Proof

This follows from Theorem 2.11 and Theorem 2.12.

Corollary 2.8 *Let (Q, \cdot) be a loop. Let $V_i(Q) = \{(Q, \cdot), (Q, \circ_i), (Q, *_i)\}$ and $S_i(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_i)\}, \{(Q, *_i)\}, \{(Q, \circ_i), (Q, *_i)\}\}$ for $i = 0, 1, 2, 3$. If (Q, \cdot) is a universal Osborn loop, then $K_{0123}(Q) = \bigcup_{i=0}^3 K_i(Q) = \left(\bigcup_{i=0}^3 V_i(Q), \bigcup_{i=0}^3 S_i(Q) \right)$ is a simplicial complex of isotopes of (Q, \cdot) .*

Proof

This is proved by combining Corollary 2.6 and Corollary 2.7.

Theorem 2.13 *Let (Q, \cdot) be a loop. Let $V_{01}(Q) = \{(Q, \cdot), (Q, \circ_0), (Q, *_0), (Q, \circ_1), (Q, *_1)\}$ and $S_{10}(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_0)\}, \{(Q, *_0)\}, \{(Q, \circ_1)\}, \{(Q, *_1)\}, \{(Q, \circ_0), (Q, *_0)\}, \{(Q, \circ_1), (Q, *_1)\}, \{(Q, \circ_0), (Q, \circ_1)\}, \{(Q, *_0), (Q, *_1)\}, \{(Q, \circ_0), (Q, *_1)\}, \{(Q, \circ_1), (Q, *_0)\}, \{(Q, \circ_0), (Q, \circ_1), (Q, *_0)\}, \{(Q, \circ_0), (Q, \circ_1), (Q, *_1)\}, \{(Q, *_0), (Q, *_1), (Q, \circ_0)\}, \{(Q, *_0), (Q, *_1), (Q, \circ_1)\}, \{(Q, \circ_0), (Q, \circ_1), (Q, *_0), (Q, *_1)\}\}$. Then, (Q, \cdot) is a universal Osborn loop and obey Equation 12 and Equation 13 if and only if $K_{10}(Q) = (V_{01}(Q), S_{10}(Q))$ is a simplicial complex of isotopes of (Q, \cdot) .*

Proof

This is proved with the aid of Theorem 2.9, Theorem 2.10, Theorem 2.6 and Theorem 2.7.

Theorem 2.14 *Let (Q, \cdot) be a universal Osborn loop. Let $V_i(Q) = \{(Q, \cdot), (Q, \circ_i), (Q, *_i)\}$, $S_i(Q) = \left\{ \{(Q, \cdot)\}, \{(Q, \circ_i)\}, \{(Q, *_i)\}, \{(Q, \circ_i), (Q, *_i)\} \right\}$ and $K_i = (V_i(Q), S_i(Q))$ for $i = 0, 1, 2, 3$. Define $f_{ij} : K_i \rightarrow K_j$ as*

$$f_{ij} : \begin{cases} (Q, \cdot) & \mapsto (Q, \cdot) \\ (Q, \circ_i) & \mapsto (Q, \circ_j) \\ (Q, *_i) & \mapsto (Q, *_j) \end{cases} \quad i, j = 0, 1, 2, 3 \text{ such that } i \neq j.$$

Then, f_{ij} is a simplicial map.

Proof

This is proved by Theorem 2.9, Theorem 2.10, Theorem 2.11 and Theorem 2.12.

Theorem 2.15 *Let (G, \cdot) and (H, \star) be two loop isotopes under the triple (A, B, C) . For $D \in \{A, B, C\}$, if $D = E_1 E_2 \cdots E_i \cdots E_n$, $E_i : G \rightarrow H$, $i = 1, \dots, n$ been bijections such that there does not exist $r \geq n$ for which $D = E_1 E_2 \cdots E_i \cdots E_r$, then the length of D , $|D| = n$ units. If $D = I$, the identity mapping, then $|D| = 0$. The length of the isotopism $(G, \cdot) \xrightarrow[\text{Isotopism}]{(A, B, C)} (H, \star)$ is giving by $|(A, B, C)| = |A| + |B| + |C|$ units. For an isotopism $(G, \cdot) \xrightarrow[\text{Isotopism}]{(A, B, C)} (H, \star)$, let the two loops (G, \cdot) and (H, \star) represent points in a 3-dimensional space and let an isotopism from (G, \cdot) to (H, \star) be a line with (G, \cdot) and (H, \star) as end-points. The set of loops $V_{01}(Q) = \{(Q, \cdot), (Q, \circ_0), (Q, *_0), (Q, \circ_1), (Q, *_1)\}$ where (Q, \cdot) is a universal Osborn loop, form a rectangular pyramid with apex (Q, \cdot) .*

Proof

We shall make use of the combined commutative diagram (7) as shown in the proof of Theorem 2.1. There are four isotopes of (Q, \cdot) as shown in the combined commutative diagram (7), namely $(Q, \circ_i), (Q, *_i)$ for $i = 0, 1$. The length of each of the isotopisms $(R_{[u \setminus (xv)]}, L_u, I), (R_{\phi_0}, L_u, I), (R_v, L_{\phi_1}, I), (R_v, L_x, I)$ is 2 units. The length of each of the isomorphisms $\gamma_0(x, u, v) = \mathbb{R}_v R_{[u \setminus (xv)]} \mathbb{L}_u L_x$ and $\gamma_1(x, u, v) = \mathbb{R}_v R_{[u \setminus (xv)]} \mathbb{L}_u L_x$ is 12 units. The length of each of the isomorphisms $\gamma_{01}^\circ = \mathbb{R}_{\phi_0(x, u, v)} R_{[u \setminus (xv)]}$ and $\gamma_{01}^* = \mathbb{L}_x L_{\phi_1(x, u, v)}$ is 6 units. Hence, the four loop isotopes $(Q, \circ_i), (Q, *_i)$ for $i = 0, 1$ of (Q, \cdot) form a rectangle. Thus, taking (Q, \cdot) as an apex and the four isotopism as lines drawn from the apex to the four vertices of the rectangle, we have a rectangular pyramid.

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